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STEADY FLOWS OF NON-NEWTONIAN FLUIDS PAST A POROUS PLATE WITH SUCTION OR INJECTION

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SUMMARY

The problem of the steady flow of three classes of non-linear fluids of the differential type past a porous plate with uniform suction or injection is studied. The flow which is studied is the counterpart of the classical 'asymptotic suction' problem, within the context of the non-Newtonian fluid models. The non-linear differential equations resulting from the balance of momentum and mass, coupled with suitable boundary conditions, are solved numerically either by a finite difference method or by a collocation method with a B-spline function basis. The manner in which the various material parameters affect the structure of the boundary layer is delineated. The issue of paucity of boundary conditions for general non-linear fluids of the differential type, and a method for augmenting the boundary conditions for a certain class of flow problems, is illustrated. A comparison is made of the numerical solutions with the solutions from a regular perturbation approach, as well as a singular perturbation.

KEY WORDS Shear-thinning Shear-thickening Apparent viscosity Normal stress difference

1. INTRODUCTION

A boundary layer flow for which a simple exact solution is available within the context of the Navier–Stokes theory is the asymptotic suction problem (cf. Reference 1, pp. 367–372). The counterpart of this problem has been studied within the context of a simple non-Newtonian fluid model, the incompressible and homogeneous fluid of grade two.[2]

As both these fluids have a constant viscosity, the boundary layers are a consequence of inertial effects, the structure being somewhat modified in the latter case by the presence of a material parameter which is a measure of the normal-stress differences. The latter study also reveals that much more complexity can be expected when considering flows of non-Newtonian fluids. While only solutions allowing suction are possible in the case of the Navier–Stokes fluid, solutions that allow for suction or blowing of an incompressible fluid of grade two are possible.

Boundary layer flows of non-Newtonian fluids have been studied by various authors[3–11] because of their technological relevance. While most of this effort has been directed at studying traditional inertial boundary layers, a recent departure has been the study of boundary layers in shear-thinning power-law fluids, by Mansutti and Rajagopal[12] and Mansutti and Pontrelli,[13]

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wherein boundary layers are found even when inertial effects are ignored. The pronounced boundary layers, by which we mean the concentration of the vorticity in a thin layer adjacent to the boundary, are due to the shear-thinning nature of the fluid, due to the non-linear dependence of viscosity on the stretching tensor.

In this paper, we plan to study the 'asymptotic suction' problem for several non-Newtonian fluids past a porous plate, subject to suction or injection at the plate, with a view to delineate the effects of the various material parameters such as density, shear-thinning (or shear-thickening) viscosity and elasticity. Such a study might reveal some interesting information on the competition between these various effects in non-Newtonian fluids. In order to do this, we pick three models. First, the power-law fluid which has a shear-dependent viscosity, but which can exhibit no normal stress differences. Next, we study the flow of a generalized power-law fluid of grade two, which has been recently used with success in modelling the flow of icy mush,[14,15] which exhibits both shear-thinning and normal stress differences. Finally, an incompressible thermodynamically compatible fluid of grade three, which has a different structure for the generalized viscosity than the power-law model, and also exhibits normal stress differences, is considered.

It would be appropriate to point out that the problem is not only of academic interest but is also of technical relevance. In the formation of polymeric or metallic sheets, the material in the liquid phase is spread over a horizontal porous sheet where it starts to solidify. The boundary layer which is formed affects the homogeneity of the sheet and to control the boundary layer thickness, the porous plate is subject to suction. Thus, the precise structure of the boundary layer is of consequence in many industrial applications. This of course presumes that the fluid under consideration can be modelled by one of the above fluid models of the differential type, and this may not be the case. Within the context of the Navier–Stokes fluid, suction is used for boundary layer control, as suction reduces the boundary layer thickness. The same is to be expected in the case of non-Newtonian fluids. However, it is possible that results contrary to those for Navier–Stokes fluids are possible in non-Newtonian fluids. For instance, while there exists no solution for blowing for the asymptotic suction problem for a Navier–Stokes fluid, solutions with blowing are possible in non-Newtonian fluids provided the material parameters satisfy certain conditions.[2]

In general, for fluids of the differential type of grade n, the equations of motion are of order $(n+1)$. Thus, if $n>1$, then the adherence boundary condition is insufficient for determinacy. The standard method used to overcome this difficulty is to resort to perturbation that lowers the order of the equation,[3–9] but this is not mathematically rigorous as a singular perturbation is treated, as though it is regular. In fact, the authors[3–9] are aware of this, but in the absence of any rational method for generating additional boundary conditions, they have no other way out of the impasse. It is possible that in flows in unbounded domains we can obtain additional conditions based on the asymptotic structure of the flow at infinity. Thus, this problem provides an opportunity to check how erroneous or correct the approach adopted in References 3–9 is. We find that solving the problem by appropriately augmenting the boundary conditions and using a singular perturbation approach, solving it numerically, and also by a regular perturbation as employed in References 3–9, the results agree remarkably well. This gives us some confidence in the methods used in References 3–9, and that may be such a method can provide useful results even when we are unable to properly augment the boundary conditions.

We integrate the equations numerically by a finite difference scheme or, in order to increase the accuracy, by a collocation method built with a B-spline function basis. For all the models considered, we determine physically meaningful solutions when suction is present at the flat plate. While we find solutions for the case of the generalized fluid of grade two, we are unable to find solutions for the power-law fluid, suggesting the need of the normal stress differences to make such solutions possible. Actually, when this quantity decreases, boundary layers assume very sharp profiles (they are almost singular at the wall). Moreover, the solutions that we find for the generalized fluid of grade two and for the fluid of grade three exhibit boundary layer thicknesses that are smaller or equal to the boundary layer thickness in the corresponding fluid of grade two, when either shear-thinning or shear-thickening occurs. We find that the shear stress is dependent on the suction/blowing velocity, on the mass density and the free stream velocity.

We also solve the problem by a perturbation method adopting either the viscosity or the elasticity modulus as a perturbation parameter, which results in a regular or a singular perturbation, respectively. In fact, just the solution to the first order, in both cases, is very close to the full numerical solution. The study also stems from our general aim to check the validity of the perturbation technique that is frequently used in non-Newtonian mechanics, namely, using a regular perturbation, and lowering the order of the equation in the light of a paucity of boundary conditions.[16–18] Moreover, this allows us to get information on the effect of the higher-order terms on the boundary layer structure.

The formulation of the equations and the solution methods are presented in the second and third sections, respectively, the final section is concerned with the description of the numerical results.

2. GOVERNING EQUATIONS

The flow past a porous plate with suction or injection and with uniform stream at infinity has a two-dimensional (2D) structure.[1] It is governed by the momentum and mass conservation laws, which are represented by the following equations:

$$\rho\, \mathrm{d}\mathbf{q}/\mathrm{d}t = \operatorname{div} \mathbf{T} + \rho \mathbf{b} \tag{1}$$

$$\operatorname{div} \mathbf{q} = 0 \tag{2}$$

with $\mathbf{q} = (u, v)$ the velocity vector, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{b}$ the specific body force, ρ the mass density and $\mathrm{d}/\mathrm{d}t$ the material derivative. We have considered three types of fluids and the expressions for their Cauchy stress tensors are the following:

power-law model

$$\mathbf{T} + p\mathbf{I} = \mu[\operatorname{tr}(\mathbf{A}_1^2)]^m\, \mathbf{A}_1 \tag{3}$$

generalized fluid of grade two

$$\mathbf{T} + p\mathbf{I} = \mu[\operatorname{tr}(\mathbf{A}_1^2)]^m\, \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 \tag{4}$$

fluid of grade three

$$\mathbf{T} + p\mathbf{I} = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta[\operatorname{tr}(\mathbf{A}_1^2)]\, \mathbf{A}_1 \tag{5}$$

where p is the indeterminate part of the stress due to the constraint of incompressibility, $\mu > 0$ is the viscosity, $\alpha_1 \geq 0$, $|\alpha_1 + \alpha_2| \leq \sqrt{(24\, \mu\beta)}$, $\beta \geq 0$ (cf. References 19–21) and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin–Ericksen tensors,[22] respectively.

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}} \tag{6}$$

$$\mathbf{A}_2 = \mathrm{d}\mathbf{A}_1/\mathrm{d}t + \mathbf{L}^{\mathrm{T}} \mathbf{A}_1 + \mathbf{A}_1 \mathbf{L} \tag{7}$$

with $\mathbf{L} = \operatorname{grad} \mathbf{q}$.

We study the steady flow of the above fluids in the absence of body forces. We choose the x-axis parallel to the plate and the y-axis normal to it, and the velocity field to depend only on y. Under this hypothesis, from equation (2) it follows that $v(y)=v_0$, v_0 being the value of the vertical velocity at the plate; v_0 is negative for suction and positive for injection.

The quantities $\mu[\mathrm{tr}(\mathbf{A}_1^2)]^m$ in equations (3) and (4), and $\mu+\beta[\mathrm{tr}(\mathbf{A}_1^2)]$ in equation (5) are called 'apparent viscosity' μ_{app}. The fluid is said to be shear-thinning (resp. shear-thickening) when $d\mu_{\mathrm{app}}/du'<0$ (resp. $d\mu_{\mathrm{app}}/du'>0$). It turns out that models (3) and (4) describe shear-thinning fluids for $m<0$ and shear thickening fluids for $m>0$ and model (5) describes only shear-thickening fluids, as we only consider $\beta>0$.

Furthermore, we observe that the above models allow non-zero normal stress difference, $T_{xx}-T_{yy}$, only for $\alpha_1\neq 0$.

As the second component of equation (1) reduces to $\partial p/\partial y=0$ and we have, for the uniformity of the free stream at infinity, $\partial p/\partial x=0$, and the equations of motion for the three models (3), (4), (5) are, respectively,

$$\rho v_0 u'=\mu[2^m(u')^{2m+1}]' \tag{8}$$

$$\rho v_0 u'=\mu[2^m(u')^{2m+1}]'+\alpha_1 v_0 u''' \tag{9}$$

$$\rho v_0 u'=\mu u''+\alpha_1 v_0 u'''+6\beta(u')^2 u''. \tag{10}$$

As boundary conditions, we impose the adherence condition at the plate

$$u(0)=0 \tag{11}$$

a free stream matching condition at infinity

$$u\to U, \quad \text{as } y\to\infty \tag{12}$$

Note that equations (9) and (10) are one order higher than the Navier–Stokes equations and thus we require an additional boundary condition.[16,17] Since the flow takes place in an unbounded domain, we can augment the boundary conditions based on the asymptotic structure of the velocity field or the boundedness of the solutions.[23] Here, we augment the boundary conditions by requiring the vanishing of the shear stress at infinity as an extra boundary condition:

$$T_{xy}\to 0 \quad \text{as } y\to\infty. \tag{13}$$

For each equation, as a first step in the solution procedure, we integrate analytically. The integration constant is computed by imposing the free stream matching condition (12).

Finally, we obtain

$$2^m(u')^{2m+1}-Ru+\sigma=0 \tag{14}$$

$$\delta u''+2^m(u')^{2m+1}-Ru+\sigma=0 \tag{15}$$

$$\delta u''+(1+\theta(u')^2)u'-Ru+\sigma=0 \tag{16}$$

where $\eta=\alpha_1/\rho$, $R=\rho v_0/\mu$, $\delta=R\eta$, $\theta=2\beta/\mu$, $\sigma=RU$.

We choose $U=1$. This is tantamount to a partial non-dimensionalization. At this juncture it would be appropriate to point-out that we shall not present a non-dimensionalized study of the equations, as our primary aim is to qualitatively delineate the effect of the various material parameters on the flow by allowing one of them to vary, the others being fixed. Each of the figures presented in the section on results and discussions can be interpreted within such a spirit.

A simple analysis leads to the conclusion that for the generalized fluid of second grade injection is possible only, if $\alpha_1\neq 0$. We first consider equation (14). Here, solutions with $u'<0$ are not admitted, i.e. we seek solutions with $u'\geq 0$ and hence $0\leq u\leq U$. Note that the term $(u')^{2m+1}$ arises as a consequence of the term $(\mathrm{tr}\,\mathbf{A}_1^2)^m\,\mathbf{A}_1$ in model (3). Since in the case of injection $R>0$, the term

$-Ru+\sigma=R(U-u)$ in equation (14) is non-negative. In fact, $-Ru+\sigma=R(U-u)$ is strictly positive for some $y\in[0, \infty)$ and thus the left-hand side of equation (14) is strictly positive while the right-hand side is zero, which implies that a solution with injection is not possible. On the other hand, in equation (15), for the case of injection we cannot apply this argument and no such conclusion can be reached. However, we have been unsuccessful in obtaining a solution in the case of injection when $\alpha_1 \neq 0$.

Finally, in order to show more clearly the dependence of the flow on the parameters α_1 (resp. β) for the third grade fluid, we also solve equation (16) by an asymptotic expansion[24] of the velocity u in power series in δ (resp. in θ/δ).

Regular perturbation method

We say $\varepsilon_1=\theta/\delta$ and we suppose that the solution u can be expanded in a power series in ε_1:

$$u=\sum_{n=0}^{\infty} \varepsilon_1^n u_n. \tag{17}$$

We observe that the perturbation parameter does not multiply the highest-order derivative. Thus, the perturbation by virtue of the boundary conditions (11)–(13) is a regular perturbation.

By substituting equation (17) into equation (16) and by equating powers of ε_1, the zeroth and first-order approximations are obtained:

$$\delta u_0''+u_0'-Ru_0+R=0 \tag{18}$$

$$\delta u_1''+u_1'-Ru_1+u_0'^3\delta=0. \tag{19}$$

We solve both the above equations with the boundary conditions (11)–(13). Note that u_0 is the solution to the second grade fluid equation (see (16) with $\theta=0$) and $\varepsilon_1 u_1$ is the first-order correction due to the third grade parameter θ.

It is trivial to show that

$$u_0(y)=1-e^{\lambda_2 y} \tag{20}$$

$$u_1(y)=\frac{\lambda_2^2}{2(3\lambda_2-\lambda_1)}\left[3^{3\lambda_2 y}-e^{\lambda_2 y}\right] \tag{21}$$

where

$$\lambda_1=\frac{1}{2}\left[-\frac{1}{\delta}+\sqrt{\left(\left(\frac{1}{\delta}\right)^2+\frac{4R}{\delta}\right)}\right] \tag{22}$$

$$\lambda_2=\frac{1}{2}\left[-\frac{1}{\delta}-\sqrt{\left(\left(\frac{1}{\delta}\right)^2+\frac{4R}{\delta}\right)}\right]. \tag{23}$$

The values for u_0, and

$$u=u_0+\varepsilon_1 u_1$$

are compared with the numerical solution and the solution for the singular perturbation in Table II.

Singular perturbation method

Now we set $\varepsilon_2=\delta$ and we suppose that the solution u can be expanded once again in a power series in ε_2. Note that the perturbation parameter multiplies the highest order derivative, and thus, we reduce the order of the equation by employing such a perturbation.

The zeroth and first-order equation are, respectively,

$$[1+\theta(u_0')^2]u_0'-Ru_0+R=0 \tag{24}$$

$$[1+3\theta(u_0')^2]u_1'-Ru_1+u_0''=0 \tag{25}$$

where $\varepsilon_2 u_1$ is the first-order correction to u_0 due to the presence of normal stress difference in the fluid (see equation (5)).

In this case, the perturbation is singular as the order of the above equations is lower than that of the original equation (16). Thus an inner and outer expansion is considered by solving equations (24) and (25) once in a neighbourhood of $y=0$ and imposing equation (11) and once at infinity by imposing equation (13). Unlike problems that usually require multiple scaling and complicated matched asymptotics, this problem is such that the outer expansion leads to the trivial solution $u_0\equiv 1$, $u_1=0$, with the inner expansion, satisfying this condition at its boundary, which is achieved by $y=10$. Physically, this suggests that the solution is smooth enough that the growth of the highest order derivative does not dominate the flow. In fact, we see that this is indeed the case as our solutions from regular perturbation and singular perturbation and the numerical solution agree very well. Solutions over the whole domain, $[0, \infty]$ are obtained by the combination of the inner and outer expansion solutions. We require that the inner solution approach the outer solution in a twice continuously differentiable manner.

In the case of singular perturbation, we once again compare our results with those for the full solution, for values of ε_2 as high as $\varepsilon_2=1$, and find the absolute error to be once again of the order of 10^{-3} (cf. Table II). The solution presented in Table II is that for the inner solution which approaches the outer solution which is the trivial identity solution.

3. NUMERICAL METHODS

First, we solved the problem by a second-order finite difference scheme,[25] for the power-law model (14). Then, in order to increase the accuracy, a collocation method with a B-splines function basis is adopted. Equations (15) and (16) are integrated up to a point $y_{\max}$, large enough that equation (13) is satisfied. In the following discussion, the main characteristic of this method is summarized.[26]

Let $Y=[y_i]_{i=1}^{l+1}$ be strictly increasing sequence of breakpoints such that $0=y_1<y_2<\cdots<y_{l+1}=y_{\max}$. Let $\sum=[\sigma_i]_{i=1}^{n}$ be a non-decreasing sequence containing y_1 and y_{l+1} $p+k$ times and each internal breakpoint k times, k, p and n being integers such that $p=2$, the order of the differential equation $k>1$ and $n=kl+p$.

The set of B-splines of order $k+p$ built on the knot sequence $\sum$, $[B_i]_{i=1}^{n}$ is a basis for the space $S_{k+p,\Sigma}=P_{k+p,Y}\cap C^{p-1}(0, y_{\max})$ where $P_{k+p,Y}$ is the set of piecewise polynomial functions of degree $k+p-1$ with breakpoint sequence Y and $C^{p-1}(0, y_{\max})$ has the usual meaning.[26]

It is known that the following properties hold:

(i) $B_i(y)>0 \quad \forall y\in[\sigma_1, \sigma_{p+n+k}], \quad i=1,\ldots,n$

(ii) $B_i(y)>0 \quad \forall y\notin[\sigma_1, \sigma_{i+k+p}], \quad i=1,\ldots,n$
which implies that the only $k+p$ B-splines are non-zero in each interval $[y_i, y_{i+1}]$;

(iii) $f(y)=\sum_{j=1}^{n} c_jB_j=\sum_{j=i-(k+p)+1}^{i} c_jB_j \qquad i=1,\ldots,p+n-k-1 \quad \forall y\in[y_i, y_{i+1}]$

(iv) $\sum_{i=r+1-(k+p)}^{s-1} B_i(y)=1 \quad \forall y\in[y_r, y_s]$

(v) $[B_i(y)]_{i=1}^n$ is a relatively well conditioned basis:

$$D_{k,\infty}^{-1}\|\bar{c}\| \le \left\| \sum_{j=1}^{n} c_j B_j(y) \right\| \le \|\bar{c}\|, \quad \bar{c}=(c_1, \ldots c_m)$$

with $D_{k,\infty}$ a positive constant depending only on k and not on the particular knot sequence Σ.

We seek solutions belonging to $S_{k+p,\Sigma}$

$$u(y)=\sum_{i=1}^{n} c_i B_i(y)$$

such that

$$G(y, u, u', u'')=0$$

where G is a shortened form for equation (15) or (16).

Let $[\tau_j]_{j=1}^{kl}$ be a sequence of collocation points selected by choosing k distinct points in each subinterval $[y_i, y_{i+1}]$.

The coefficients c_i are computed as solutions of the non-linear algebraic system:

$$G\left(\tau_j, \sum_{i=1}^{n} c_i B_i^2\right)=0 \quad j=1, \ldots, kl \quad s=0, 1, 2 \tag{26}$$

with the discretized form for equations (11) and (13).

It is known that the accuracy provided by a collocation method can be increased by choosing the roots of kth Legendre polynomial as the collocation points; the global error is $O(|\Delta|^{p+k})$, $|\Delta|$ being the maximum subinterval length.[27]

At the breakpoints, the order of the approximation is $O(|\Delta|^{2k})$. The value of k and l depend on the approximating function, and in our computations we fixed $k=4$ and chose a variable number of breakpoints, more concentrated where strong gradients in the solution are expected.

In the numerical experiments, we let all parameters present in the model vary, in order to gain insight on the role played in the corresponding flow by each of them. We solved equation (26) subject to the appropriate boundary conditions by the Powell method and locally parameterized continuation method[28] applied along one parameter at a time which will be referred to as PAR. When possible, we initialized with the solution for the Navier–Stokes or second grade fluid and we proceeded with PAR. The numerical code was built in a vectorized form and run on the IBM 3090 at CICS in Rome.

Table I. Analytical solutions for the fluid flow

y_{max} (suction)	y_{max} (blowing)	abs (R)
71·79	0·1	0·1
10·67	0·9	1
6·53	1·58	2
4·97	2·32	4
4·52	2·59	8
4·52	3·74	20
4·52	4·11	50
4·52	4·52	100
4·52	4·97	500

The following results have been obtained by collocation. For $m=0$, the analytical solution (cf. Table I) for the fluid of second grade has been used to check the accuracy of the numerical scheme. We have computed the solution for several values of the normal stress modulus α_1 and/or the suction or blowing velocity and, by using 20 breakpoints (with an automatic local redistribution) and B-spline functions of degree five, an absolute error of $O(10^{-8})$ has been obtained. For the fluid of second grade, the thickness of the boundary layer decreases for increasing values of the suction velocity and increases for increasing values of the blowing velocity.

4. RESULTS AND DISCUSSION

Before we discuss the numerical results, we provide below a comparison of the results from the regular and singular perturbation with the numerical solution for the case $R=-1$, $\theta=0{\cdot}1$, $\delta=-0{\cdot}1$, as given in Table II. We see that the results agree well.

The first set of numerical results that we describe is related to the solution to the power-law fluid model (equation (14)).

In this case the equation has been solved by a finite difference scheme using 100 grid points for sufficient accuracy; for the case $m=0$, matching with the Navier–Stokes analytical solution provides a relative error $<0{\cdot}0005$. Figures 1 and 2 show the u-profiles for varying m corresponding to shear thinning fluids for two fixed values of the suction velocity. It appears that when the suction velocity is strong enough, it enhances shear thinning effects. Increasing the suction velocity decreases the boundary layer thickness (Figure 2). Figure 3 shows that the change in behaviour of the solution with respect to the suction velocity.

Table II. Comparison of the numerical solution and the perturbations

		Regular perturbation		Singular perturbation	
y	u	u_0	u	u_0	u
0·335	0·254793	0·264263	0·253900	0·269839	0·254269
0·835	0·524017	0·534630	0·523434	0·550231	0·521983
1·335	0·697696	0·705643	0·697387	0·725510	0·695409
1·835	0·808466	0·813813	0·808293	0·833122	0·806647
2·335	0·878769	0·882232	0·878666	0·898694	0·877587
2·835	0·923298	0·925509	0·923235	0·938535	0·922660
3·335	0·951479	0·952883	0·951439	0·962715	0·951222
3·835	0·969308	0·970197	0·969283	0·977385	0·969282
4·335	0·980586	0·981149	0·980571	0·986283	0·980683
4·835	0·987720	0·988076	0·987710	0·991680	0·987867
5·335	0·992233	0·992458	0·992227	0·994954	0·992389
5·835	0·995087	0·995230	0·995083	0·996939	0·995231
6·335	0·996892	0·996983	0·996890	0·998144	0·997014
6·835	0·998034	0·998091	0·998033	0·998874	0·998133
7·335	0·998757	0·998793	0·998756	0·999317	0·998833
7·835	0·999214	0·999236	0·999213	0·999586	0·999272
8·335	0·999503	0·999517	0·999502	0·999749	0·999546
8·835	0·999685	0·999695	0·999685	0·999848	0·999717
9·335	0·999801	0·998807	0·998801	0·999908	0·999824
9·835	0·999933	0·999935	0·999933	0·999944	0·999890

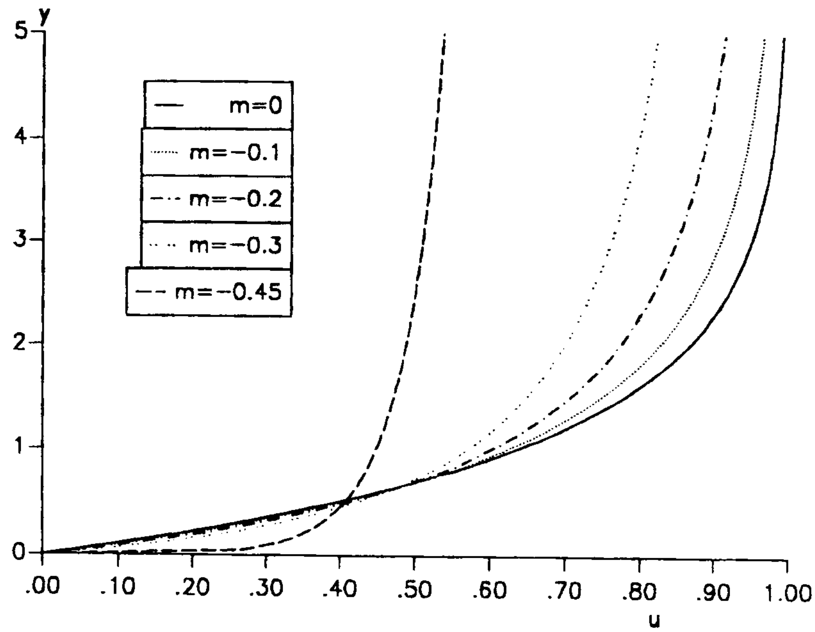

Figure 1. Velocity profiles for equation (14). Suction case: $R=-0{\cdot}5$

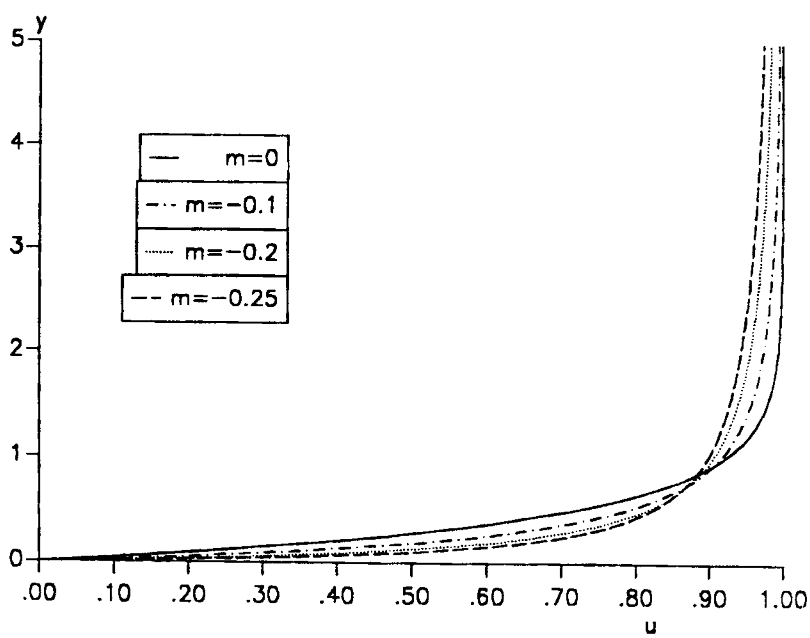

Figure 2. Velocity profiles for equation (14). Suction case: $R=-1$

When shear-thickening effects are considered, solutions have the shape displayed in Figure 4, where it can be seen that, when m increases, suction induces sharper boundary layers.

Results described above report solutions up to $y=5$, where the free stream velocity has not yet been reached in most cases. Actually, in order to reach that point (assumed to be infinity) a stretching transformation has been introduced. We adopted: $y^* = \ln[(a-x_{max})/y^*_{max}y + x_{max}]$, where a and x_{max} are parameters such that $0<a<1$ and $x_{max}>1$. By using $n=100$ grid points and

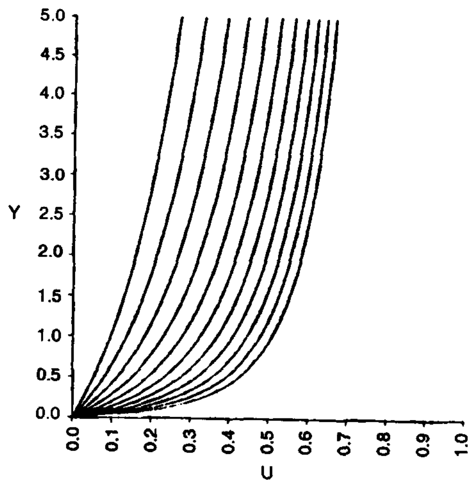

Figure 3. Velocity profiles for equation (14). Suction case: $m=-0{\cdot}4$, $-1<R-0{\cdot}5$, $\Delta R=0{\cdot}05$

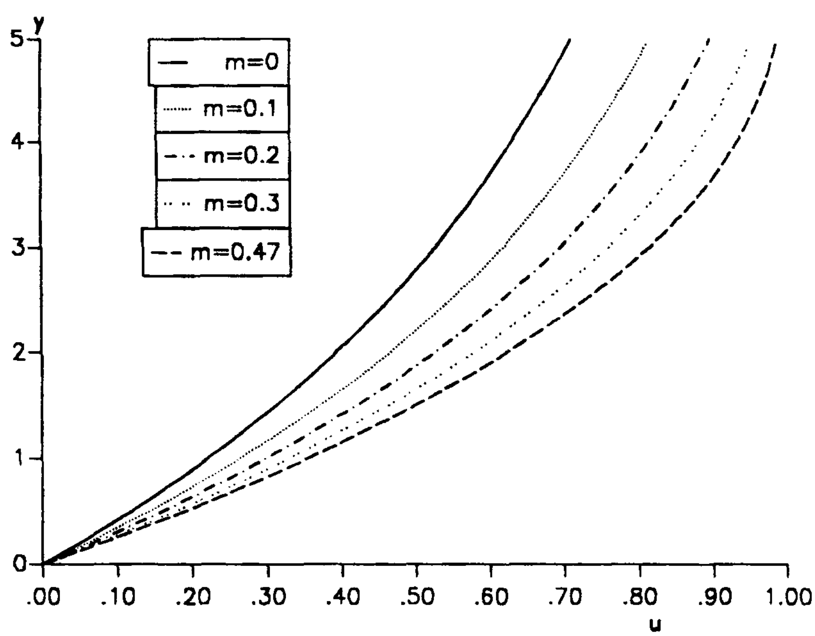

Figure 4. Velocity profiles for equation (14). Suction case: $R=-0{\cdot}25$

for suction velocity $R=-1$ and $m=-0{\cdot}45$, u reached the value 0·855 at $y=1000$, and thus the free stream velocity 1 has not yet been achieved.

In the case of the generalized power-law fluid of second grade, normal stress differences add to the shear-thinning and shear-thickening effects and, as a consequence, solutions are possible even in the case of injection.

Either with blowing or with suction, for increasing values of δ, the thickness of the boundary layer increases so that in general we can say that normal stress differences counteract the formation of such a structure. This is apparent in Figure 5 where results in the blowing case are reported, showing smoother profiles in the vicinity of the plate as δ increases.

Figures 6 and 7 displays the velocity field u, for values of m, with suction at the plate, for shear-thickening (shear-thinning) fluid from which the effects of the normal stress differences can be determined.

We found that in the case of injection, the velocity profiles were not that sensitive to variations in the index m, at least for the range $0 \leq m \leq 0{\cdot}4$ (recall that $m=0$ corresponds to the fluid of grade two). It is worth noting that the difference in the order of magnitude of the boundary layer thickness in the case of suction and injection (see also the above table).

As far as the third grade fluid model (16) is concerned, for both suction and blowing, we present two sets of results, for variations of the apparent viscosity parameter θ, and of R. We find that in the case of suction, decreasing the values of θ (Figure 8) causes a sharpening of the boundary layer, as in the case of the power-law fluid (cf. Figure 4). Also, increasing the values of the suction velocity, provides a sharper boundary layer, as shown in Figure 9. Finally, increasing the rate of injection increases the thickness of the boundary layer as evidenced in Figure 10.

We conclude by making a few remarks about the perturbation solutions. The singular perturbation method is based on the choice of δ as the perturbation parameter and yields equations independent of η, and thus it allows for a better understanding of the effect of the

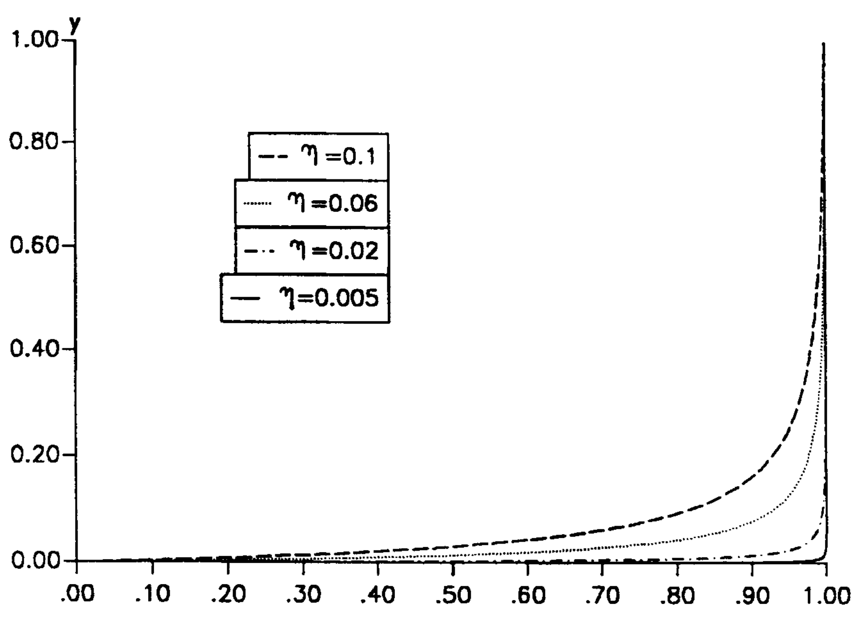

Figure 5. Velocity profiles for equation (15). Blowing case: $R=1$, $m=0{\cdot}2$

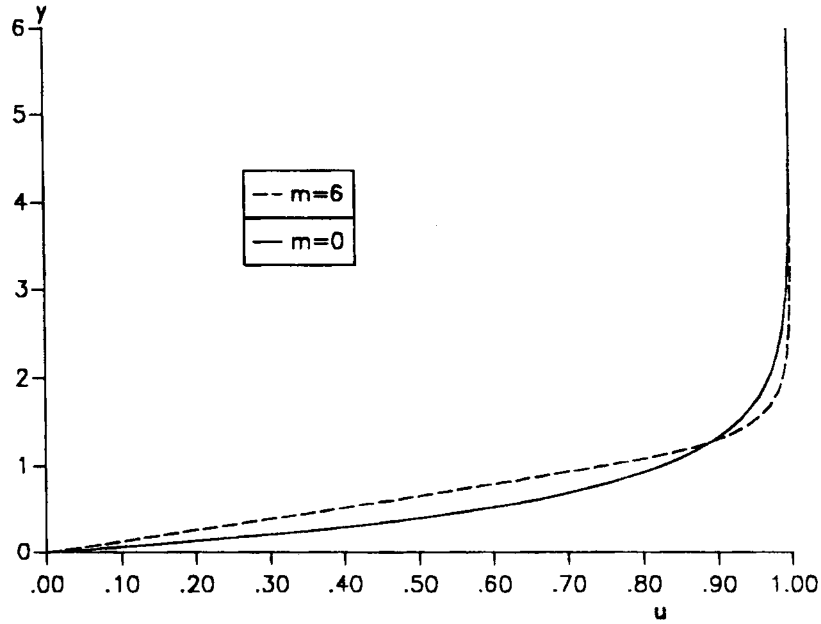

Figure 6. Velocity profiles for equation (15). Suction case: $R=-2{\cdot}5$, $\delta=-0{\cdot}25$

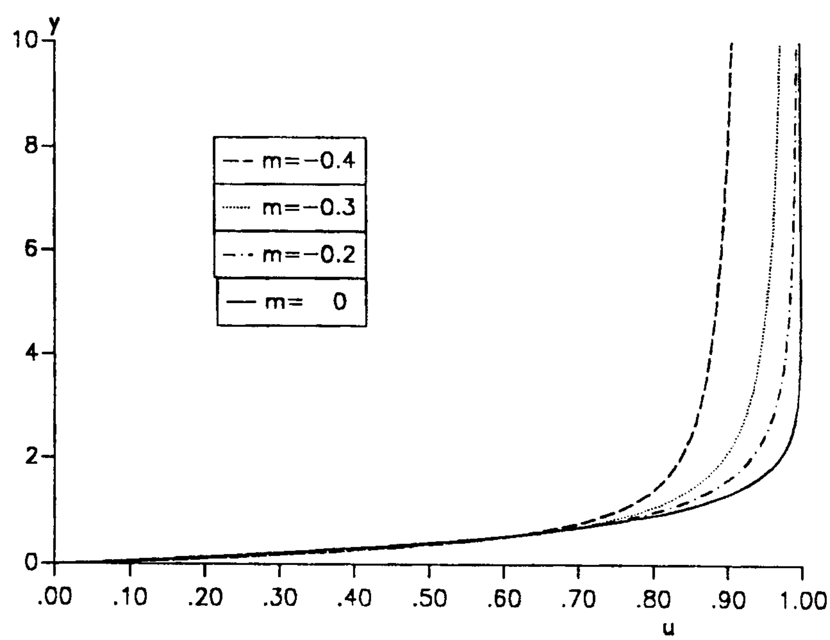

Figure 7. Velocity profiles for equation (15). Suction case: $R=-2{\cdot}5$, $\delta=-0{\cdot}25$

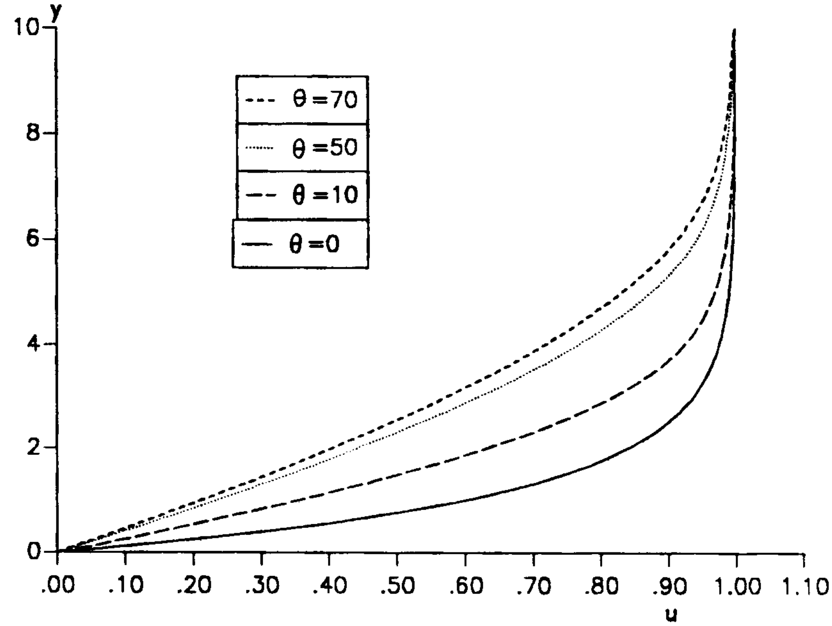

Figure 8. Velocity profiles for equation (16). Suction case: $R=-1$, $\delta=-0{\cdot}1$

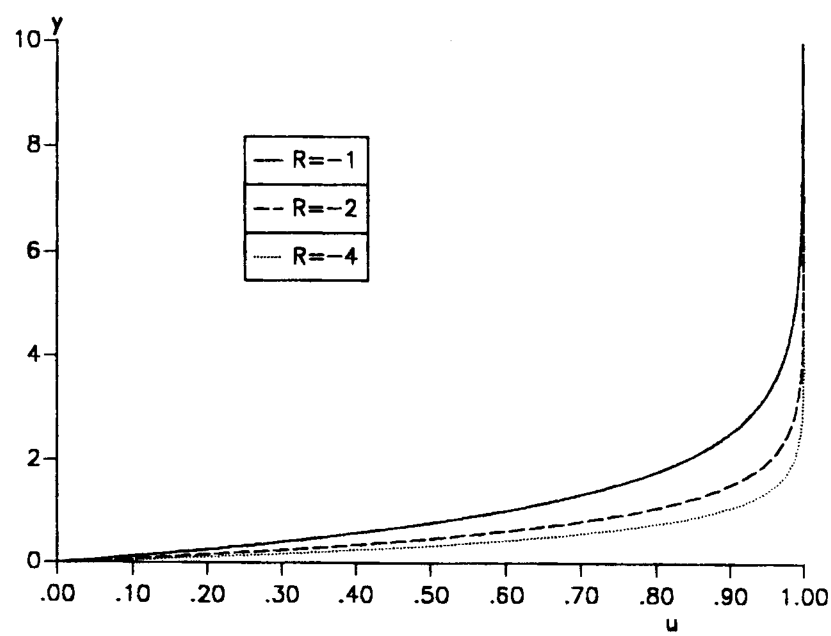

Figure 9. Velocity profiles for equation (16). Suction case: $\theta=0{\cdot}1$, $\delta=0{\cdot}1*R$

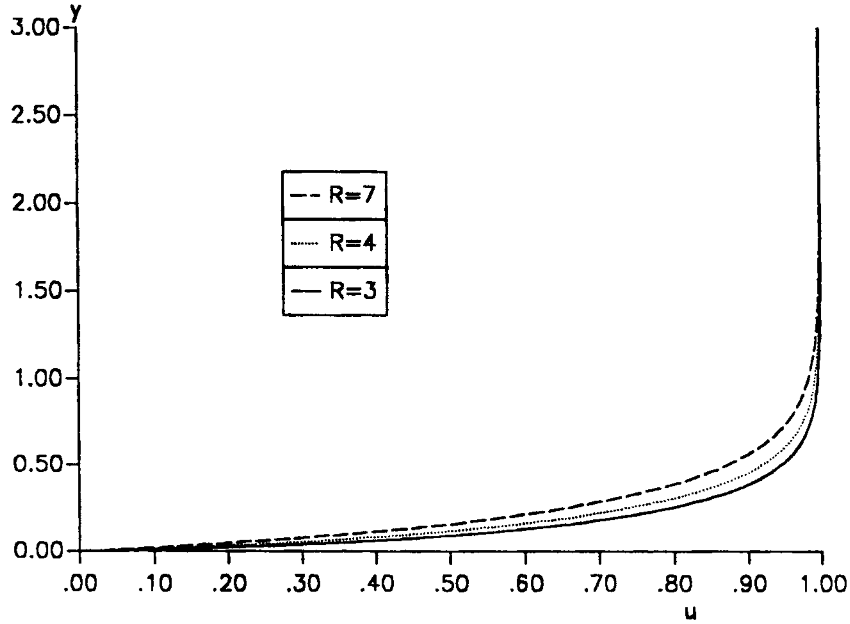

Figure 10. Velocity profiles for equation (16). Blowing case: $\theta=0{\cdot}1$, $\delta=0{\cdot}1*R$

solution on the specific perturbation parameter for various values of η. We find that increasing the normal stress coefficient leads to an increase in the boundary layer thickness. Here, computations have been performed for $\theta=0{\cdot}1$, $R=-1$ and $\eta=0{\cdot}001, 0{\cdot}01, 0{\cdot}1$ ($\varepsilon_2=-0{\cdot}001$, $-0{\cdot}01$, $-0{\cdot}1$). Similarly, in the case of regular perturbation, the coefficient θ has been used as the perturbation parameter and the zeroth and first-order equations are independent of θ. Results for $\eta=0{\cdot}1$, $R=-1$ and $\theta=0{\cdot}001, 0{\cdot}01, 0{\cdot}1$ ($\varepsilon_1=-0{\cdot}01$, $-0{\cdot}1$, -1), clearly confirm (see also Figure 10) that the more fluid shear thickens the thicker the boundary layer.

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